$(2/3)(a+b+c) < u < a+b+c$(Posted)

https://www.linkedin.com/groups/8313943/8313943-6368766061759860737

Let $a, b, c > 0$. Show that the equation

$x^3 - (a^2+b^2+c^2)x - 2abc = 0$

has a unique positive root u which satisfies

$\dfrac{2(a+b+c)}{3} < u < a+b+c.$

Solution by Arkady Alt , San Jose, California, USA.

Let $\varphi(x) := 1 - \dfrac{2abc}{x^3} - \dfrac{a^2+b^2+c^2}{x^2}, x \in \mathbb{R}\setminus\{0\}$and $F(x) := x^3 - (a^2+b^2+c^2)x - 2abc$.

Since $x = 0$ isn't solution of the equation $F(x) = 0$ then $F(x) = 0 \iff \varphi(x) = 0$ for any $x \neq 0$.

Since $\varphi\left(\sqrt{a^2+b^2+c^2}\right) < 0, \lim\limits_{x\to\infty} \varphi(x) = 1$ and function $\varphi(x)$ is continuous and increasing

on $(0,\infty)$ then equation $F(x) = 0$ has a unique positive root $u \in \left(\sqrt{a^2+b^2+c^2}, \infty\right)$.

Noting that in case $a = b = c$ equation $F(x) = 0$ becomes $x^3 - 3a^2x - 2a^3 = 0 \iff (x-2a)(a+x)^2 = 0$

we will prove that $u \in \left[\dfrac{2s}{3}, s\right)$, where $s := a+b+c$. Let $p := ab+bc+ca, q := abc$.

Since by AM-GM inequality $s \geq 3q^{1/3}, p \geq 3q^{2/3}$ then $F(s) = s^3 - \left(s^2-2p\right)s - 2q =$

$2(ps-q) \geq 2(9q-q) = 16q > 0$.

Also, since $9q \geq 4ps - s^3$ (Schure Inequality $\sum_{cyc} a(a-b)(a-c) \geq 0, a, b, c > 0$ in $s, p, q-$notation)

and $s^2 \geq 3p$ then $27F\left(\dfrac{2s}{3}\right) = 27\left(\left(\dfrac{2s}{3}\right)^3 - \left(s^2-2p\right)\cdot\dfrac{2s}{3} - 2q\right) = 2\left(18ps - 27q - 5s^3\right) =$

$-2\left(2s\left(s^2-3p\right) + 3\left(9q - 4ps + s^3\right)\right) \leq 0$.

Note in the latter inequality equality holds iff $s^2 = 3p \iff a = b = c$.

Thus, $\dfrac{2(a+b+c)}{3} \leq u < a+b+c$ and equality holds iff $a = b = c$.

Remark.

Additional information about cubic equation $x^3 - (a^2+b^2+c^2)x - 2abc = 0$, which has

relation to geometry, you can find by the followig links:

1. Article "Independent representation of an acute triangle with application to

inequalities",Arkady Alt, Mathemathical Reflections n.4,2009.

http://www.equationroom.com/Publications/Mathematical%20Reflections/Independent%20parametrization%20of%20an%20acute%20triangle%20and%20it's%20application-%20n.4,2009.pdf

2. Problem 5337 two solutions, SSMA,Problem section.

http://ssma.play-cello.com/wp-content/uploads/2016/03/May-2015.pdf

3. Problem 3395 with solution, CRUX vol.35,n.8

https://cms.math.ca/crux/v35/n8/page520-535.pdf.
4. Problem11443 with solution in American Mathematical Monthly Vol.118,n.3,March 2011,